Conversion of stability in systems close to a Hopf bifurcation by time-delayed coupling

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We propose a control method with time delayed coupling which makes it possible to convert the stability features of systems close to a Hopf bifurcation. We consider two delay-coupled normal forms for Hopf bifurcation and demonstrate the conversion of stability, i.e., an interchange between the sub- and supercritical Hopf bifurcation. The control system provides us with an unified method for stabilizing both the unstable periodic orbit and the unstable steady state and reveals typical effects like amplitude death and phase locking. The main method and the results are applicable to a wide class of systems showing Hopf bifurcations, for example, the Van der Pol oscillator. The analytical theory is supported by numerical simulations of two delay-coupled Van der Pol oscillators, which show good agreement with the theory.

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I. INTRODUCTION

Time delayed feedback control (DFC) [1] is a simple and convenient method to stabilize unstable periodic orbits (UPOs) occurring in time-continuous nonlinear dynamical systems. Since DFC uses only the difference of the current and the delayed state where the time delay is given by the period of the UPO, the control is non-invasive and is applicable to systems whose equations of motion are unknown. Due to this convenience, the algorithm of DFC has been applied to quite diverse experimental systems in physics, chemistry, biology, and medicine [2–9] and theoretical advances have also been made, e.g., [10–17]. However, it has been contended [18,19] that the method fails in the case of torsion-free UPOs or, more precisely, for UPOs with an odd number of real positive Floquet exponents. In order to overcome the odd number limitation, an unstable delayed feedback controller [20,21] and a half-period delayed feedback control [22] were suggested. More recently, it has been shown that the odd number limitation does not hold if the control force is coupled to the system by a rotational matrix with suitable phase [23]. The problem of stabilizing an UPO with an unstable controller in a dynamical system close to a subcritical Hopf bifurcation can be treated analytically by means of standard asymptotic methods developed in the theory of weakly nonlinear oscillators [24], but the control terms should be nonlinear, which results in a limitation of the basin of attraction to a narrow range around the UPO.

In parallel to the control of UPOs, the stabilization of unstable steady states (USSs) has become a field of increasing interest. One of the methods to control an USS introduced by Pyragas *et al.* uses the difference between the current state and a low-pass filtered version [25,26]. A DFC scheme in a diagonal coupling form, which was originally invented to control UPOs, has been analytically investigated by using the Lambert function for the purpose of stabilizing the USS [27,28]. Diffusively coupled limit cycle oscillators without time delay exhibit complex phenomena such as synchronization and amplitude death [29–32]. For weakly coupled oscillators the predominant effect is a synchronization of the frequencies of the individual oscillators to a single common frequency once the coupling strength exceeds a certain threshold, while the amplitudes remain unaffected. For stronger couplings the amplitudes also play an important role and give rise to coupling induced stabilization of an USS, that is, amplitude death of the oscillations. On the other hand, the delay induced death discovered by Ramana Reddy *et al.* [33,34], has attracted considerable interest [35] and it has been thoroughly investigated using experimental and theoretical approaches [36–39]. Two limit cycle oscillators that interact with each other via diffusive delayed coupling show amplitude death of the oscillators even if they have the same frequency, which is in sharp contrast to the situation with no time delay where amplitude death can occur only if the frequencies are sufficiently different. However, in both cases the coupling between the oscillators should be sufficiently strong for amplitude death.

The purpose of this paper is to propose a unified time-delay control method which makes it possible to stabilize either an UPO or an USS that appears in a dynamical system close to a subcritical or a supercritical Hopf bifurcation, respectively. We consider a delay-coupled system of the normal form for a Hopf bifurcation which is also known as a Stuart-Landau oscillator. In our system, the control terms do not only include the difference between the current and the delayed value but also a state variable of the controller alone, in a form that is neither diagonal nor diffusive. Therefore, the stabilization of UPOs becomes possible even under an odd number condition and also with half-period control. Since we consider a situation close to the Hopf bifurcation, all relevant features, including the criteria for phase synchronization as well as an estimate of the Floquet exponents, can be treated analytically. As a result of this we can show that the stability features close to the Hopf bifurcation are converted, resulting in the stabilization of the UPO and amplitude death. The basin of attraction for stabilizing UPO and USS has a global characteristic due to the linear coupling and the amplitude death occurs with a small coupling strength. The same results

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can be obtained also for other systems exhibiting Hopf bifurcations, for example, two delay-coupled Van der Pol type systems. This example is confirmed numerically.

The paper is organized as follows. In Sec. II we present our model and derive equations for amplitude and phase difference. Section III is devoted to the analysis of the phase difference and the phase-locking condition for two delay-coupled normal forms. In Sec. IV, we analyze the amplitude equations and demonstrate the conversion effect of stability for full-period and half-period coupling. In Sec. V, we consider two delay-coupled Van der Pol systems as another model for the Hopf bifurcation and show the conversion of stability by means of approximate analysis as well as direct numerical simulations. Finally, in Sec. VI, we draw some conclusions.

II. A MODEL OF TWO DELAY-COUPLED LIMIT CYCLE SYSTEMS

We consider the following model of a system with a limit cycle (Z_0) coupled to a control system (Z_c):

$$\dot{Z}_0(t) = (\pm\lambda_0 + i\omega_0 \mp |Z_0(t)|^2)Z_0(t) + KZ_c(t), \tag{1}$$

$$\dot{Z}_c(t) = (\lambda_c + i\omega_c - |Z_c(t)|^2)Z_c(t) - K[Z_0(t) - Z_0(t-\tau)], \tag{2}$$

where $Z_0(t)$ and $Z_c(t)$ are the complex amplitudes of the limit cycle system and of the controller, respectively. In the absence of coupling ($K=0$), Eq. (1) has a stable or unstable limit cycle at $|Z_0| = \sqrt{\lambda_0}$ for $\lambda_0 > 0$ with natural frequency ω_0 according to the choice of sign on the right hand side. That is, the system to be controlled is the normal form of the super- or subcritical Hopf bifurcation model with bifurcation parameter λ_0. Equation (2), which denotes the controller, has an unstable fixed point at the origin and a stable limit cycle at $|Z_c| = \sqrt{\lambda_c}$ with natural frequency ω_c for $\lambda_c > 0$. $K(>0)$ is the coupling strength and the delay time τ is chosen as the period of the limit cycle of the system ($\tau=2\pi/\omega_0$). If a trajectory of the system described by Eq. (1) coincides with an USS or an UPO with period τ and the controller described by Eq. (2) remains on the USS at the origin, the coupling terms vanish in Eqs. (1) and (2).Therefore, our delay-coupling method allows for noninvasive control of dynamical systems. Introducing the phases φ, ψ and the real amplitudes r, w by $Z_0(t)=r(t)e^{i\varphi(t)}$ and $Z_c(t)=w(t)e^{i\psi(t)}$, and substituting into Eqs. (1) and (2) we obtain the following equations:

$$\dot{r} = f_\pm(r) + Kw\cos(\varphi - \psi), \tag{3}$$

$$\dot{\varphi} = \omega_0 - \frac{Kw}{r}\sin(\varphi - \psi), \tag{4}$$

$$\dot{w} = (\lambda_c - w^2)w - K[r\cos(\varphi - \psi) - r_\tau\cos(\varphi_\tau - \psi)], \tag{5}$$

$$\dot{\psi} = \omega_c - \frac{K}{w}[r\sin(\varphi - \psi) - r_\tau\sin(\varphi_\tau - \psi)], \tag{6}$$

where $r_\tau \equiv r(t-\tau)$, $\varphi_\tau \equiv \varphi(t-\tau)$ and $f_\pm(r) = \pm(\lambda_0 - r^2)r$, representing the super- and subcritical Hopf normal form according to the choice of sign "+" and "–", respectively. Note that $f_+(r) = -f_-(r)$.

In the following we consider the limit cycle system close to the Hopf bifurcation, i.e., $|\lambda_0| \ll \omega_0$, and in the controller we fix the parameter λ_c at a small positive value ($0 < \lambda_c \ll \omega_c$). Then we can assume that the amplitudes $r(t)$ and $w(t)$ vary very slowly in comparison with the phases $\varphi(t)$ and $\psi(t)$:

$$\frac{|r - r_\tau|}{r} \ll 1. \tag{7}$$

Moreover, if the difference of frequencies between the system and the controller is assumed to be small ($\omega_0 \approx \omega_c$), the phase difference $\theta = \varphi - \psi$ is also a slowly varying quantity. Now the derivative of the phase φ of Eq. (4) can be regarded approximately as constant. Thus the phase φ becomes a linear function of time t, and the delayed phase φ_τ can be written as

$$\varphi_\tau \approx \varphi - \tau\dot{\varphi} = \varphi - \tau\left[\omega_0 - \frac{Kw}{r}\sin(\varphi - \psi)\right],$$

which yields

$$\varphi_\tau - \psi \approx \theta + \frac{K\tau w}{r}\sin\theta - \omega_0\tau.$$

Therefore, Eqs. (3)–(6) can be reduced to three equations for the amplitudes and the phase difference as follows:

$$\dot{r} = f_\pm(r) + Kw\cos\theta, \tag{8}$$

$$\dot{w} = \lambda_c w - K\left[r\cos\theta - r_\tau\cos\left(\theta + \frac{K\tau w}{r}\sin\theta\right)\right], \tag{9}$$

$$\dot{\theta} = \Delta\omega + K\left[\left(\frac{r}{w} - \frac{w}{r}\right)\sin\theta - \frac{r_\tau}{w}\sin\left(\theta + \frac{K\tau w}{r}\sin\theta\right)\right], \tag{10}$$

where $\Delta\omega = \omega_0 - \omega_c$ is the frequency detuning between the system and the controller. Here the cubic term of w was neglected in Eq. (9) since we confine ourselves to the behavior of the controller close to the unstable fixed point at zero.

III. ANALYSIS OF THE PHASE DIFFERENCE EQUATION: PHASE-LOCKING

By inspecting Eq. (10), we see that if $\Delta\omega=0$ then there are two stationary values $\theta_1^*=0$, $\theta_2^*=\pi$ for the phase difference. For these values Eqs. (8) and (9) have two stationary invariant solutions $(w^*, r^*) = (0,0)$ and $(w^*, r^*) = (0, \sqrt{\lambda_0})$. However, even in the case of $\Delta\omega=0$ it is difficult to evaluate directly the stability of the invariant sets of the coupled system. So we have to perform some approximation as follows. In the vicinity of the fixed point satisfying $\sin\theta^*\approx 0$, Eq. (9) can be written as $\dot{w}=\lambda_c w-K(r-r_\tau)\cos\theta$, and we can evaluate approximately the order of $w(t)$ as

$$w(t)\approx K|(r-r_\tau)\cos\theta^*|/\lambda_c$$

since the variable $w(t)$ is slow. Taking into account Eq. (7), we obtain the following estimate:

$$\frac{w}{r}\sim\frac{K}{\lambda_c}\frac{|r-r_\tau|}{r}\ll 1. \tag{11}$$

Then we can expand the second term inside the square bracket of Eq. (10) into a power series with respect to $K\tau(w/r)\sin\theta$ up to the first order as

$$\sin\left(\theta+K\tau\frac{w}{r}\sin\theta\right)\approx\sin\theta+K\tau\frac{w}{r}\cos\theta\sin\theta,$$

and Eq. (10) can be written as

$$\dot{\theta}=\Delta\omega+K\left(\frac{r-r_\tau}{w}-\frac{K\tau r_\tau}{r}\cos\theta-\frac{w}{r}\right)\sin\theta. \tag{12}$$

For $\tau|\dot{r}|/r\ll 1$, the delay term r_τ can be approximated by the first derivative as $r_\tau=r(t-\tau)\approx r(t)-\tau\dot{r}$, i.e.,

$$r-r_\tau\approx\tau\dot{r}. \tag{13}$$

By using Eq. (8) we get the expression $r-r_\tau\approx\tau[f_\pm(r)+Kw\cos\theta]$, and thus Eq. (12) yields

$$\dot{\theta}=\Delta\omega+K\left[\frac{\tau f_\pm(r)}{w}+K\tau\frac{r-r_\tau}{r}\cos\theta-\frac{w}{r}\right]\sin\theta. \tag{14}$$

This system can be simplified even further. Taking into account Eq. (7) and Eq. (11), we can neglect the second and third term in the square bracket of Eq. (14) and obtain finally the approximate equation for the phase difference as

$$\dot{\theta}=\Delta\omega+\frac{K\tau f_\pm(r)}{w}\sin\theta. \tag{15}$$

Figure 1 illustrates how the stationary points θ^* for the phase difference are positioned as intersections of the straight line of $-\Delta\omega$ and two sine curves. If

$$K\tau>w\left|\frac{\Delta\omega}{f_\pm(r)}\right|,$$

there are two stationary values in the interval $[-\pi/2,3\pi/2]$:

$$\theta^*_{\pm 1}=\sin^{-1}\left(\frac{-w\Delta\omega}{K\tau f_\pm(r)}\right),$$

$$\theta^*_{\pm 2}=\pi-\sin^{-1}\left(\frac{-w\Delta\omega}{K\tau f_\pm(r)}\right).$$

It is obvious that inside the limit cycle the function $f_+(r)$ corresponding to the supercritical Hopf bifurcation is positive, while $f_-(r)$ for the subcritical one is negative. Taking this into account, we can see from Eq. (15) that, for the USS system (supercritical), the fixed point θ^*_{+1} is unstable and θ^*_{+2} is stable. On the other hand, for the UPO system (subcritical), the fixed point θ^*_{-1} is stable and θ^*_{-2} is unstable. For example, if $\Delta\omega=0$, the UPO system and the controller are synchronized without phase difference, while the USS system and the controller are oscillating in antiphase (antisynchronization).

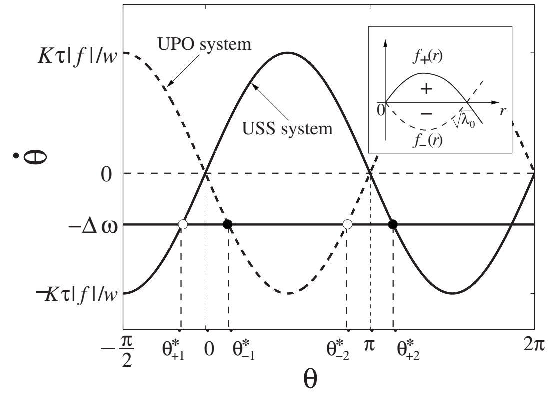

FIG. 1. Stability of the stationary points of phase difference for the USS (supercritical) and UPO (subcritical) system. From Eq. (15), the stationary points are given by intersection of the straight line of $-\Delta\omega$ and the solid (USS system) or dotted (UPO system) sine curve. The black dots and open circles on the intersection denote the stable and unstable stationary points, respectively. The inset shows the positive and negative features of the function $f_+(r)$ and $f_-(r)$ inside the limit cycle, respectively, which results in the converted relation between the solid and dotted sine curves in the main figure.

The system and the controller run at a constant frequency given by

$$\omega^*=\dot{\varphi}=\dot{\psi}=\omega_0+\frac{w^2\Delta\omega}{\tau r f_\pm(r)}.$$

This common frequency is generally not the arithmetic mean; instead the frequencies of the system and controller are shifted from the synchronized frequency by an amount determined from

$$\left|\frac{\omega_0-\omega^*}{\omega_c-\omega^*}\right|=\left|\frac{\gamma}{1+\gamma}\right|\ll 1,$$

where $\gamma=w^2/\tau r f_\pm(r)$, and away from the limit cycle $r=\sqrt{\lambda_0}$, $|\gamma|\sim(w/r)^2$ is small by Eq. (11). This means that the synchronized frequency is much closer to the frequency of the system than of the controller. If

$$K\tau<w\left|\frac{\Delta\omega}{f_\pm(r)}\right|,$$

there is no stationary phase difference, i.e., no phase locking.

IV. ANALYSIS OF AMPLITUDE EQUATIONS: CONVERSION OF STABILITY

Now, let us consider the amplitude behavior on the basis of the phase locking characteristics demonstrated above. If

$\Delta\omega \approx 0$ and the system is apart from the limit cycle, the following estimate can be seen from Eqs. (8) and (15):

$$\frac{dr}{r\,d\theta} \approx \frac{1}{K\tau \sin\theta}\frac{w}{r} + \gamma \cot\theta \ll 1.$$

Thus the variation of the phase difference before arriving at the phase locking is very fast in comparison with the amplitudes. This means that phase locking precedes the evolution of amplitudes and we are allowed to consider the amplitude equations (8) and (9) after replacing the relative phase θ by the stationary values θ^*. If $\Delta\omega = 0$, for example, the amplitude equations for the USS ($\theta^* = \pi$) and the UPO ($\theta^* = 0$) system can be reduced to

$$\dot{r} = f_+(r) - Kw,$$

$$\dot{w} = \lambda_c w + K(r - r_\tau),$$

and

$$\dot{r} = f_-(r) + Kw,$$

$$\dot{w} = \lambda_c w - K(r - r_\tau),$$

respectively. If $\Delta\omega \neq 0$ but $\sin\theta^* \approx 0$, more generally, Eq. (9) can be written as $\dot{w} = \lambda_c w - K(r - r_\tau)\cos\theta^*$, which yields the final amplitude equations as follows:

$$\dot{r} = f_\pm(r) + \hat{w}, \tag{16}$$

$$\dot{\hat{w}} = \lambda_c \hat{w} - k(r - r_\tau), \tag{17}$$

where $\hat{w} = Kw\cos\theta^*$ and $k = K^2\cos^2\theta^*$. It is obvious that Eqs. (16) and (17) have fixed points at $(r^*, \hat{w}^*) = (r_0, 0)$, where $r_0 = 0$ or $r_0 = \sqrt{\lambda_0}$. By using approximation (13), the time-delay system (16) and (17) can be transformed to a system of ordinary differential equations:

$$\dot{r} = f_\pm(r) + \hat{w}, \tag{18}$$

$$\dot{\hat{w}} = \lambda_c \hat{w} - k\tau\dot{r}, \tag{19}$$

which is, in essential, equivalent to the low pass filtering control system considered in [25]. Since Eq. (19) describes a high pass filter with a slow input signal $r(t)$ it follows naturally that the output signal $w(t)$ is very small in comparison with $r(t)$, which was derived earlier in Eq. (11). We can rewrite Eqs. (18) and (19) as

$$\ddot{r} + [k\tau - f'_\pm(r) - \lambda_c]\dot{r} + \lambda_c f_\pm(r) = 0,$$

which yields, by using the slow time-scale of the variable $r(t)$, the approximation: $[k\tau - f'_\pm(r) - \lambda_c]\dot{r} = -\lambda_c f_\pm(r)$. Then we can see

$$\dot{r} \sim -f_\pm(r) = f_\mp(r), \tag{20}$$

if the following conditions are fulfilled:

$$\lambda_c > 0, \quad k\tau > f'_\pm(r_0) + \lambda_c. \tag{21}$$

Note that the system without the delay control is described by $\dot{r} = f_\pm(r)$, and compare it with Eq. (20) that denotes the effective system with the control. We see that the stability of the controlled system has been converted, i.e., the stable fixed point becomes unstable and the unstable fixed point becomes stable in the amplitude equations. In the limit cycle system, Eq. (20) implies that the UPO and the USS have been transformed to a stable limit cycle and a stable fixed point, respectively, and vice versa. This means that stabilization of the UPO and USS might be efficiently achieved, that is, the stability of the Hopf bifurcation system might be strikingly converted by the time-delay coupling with a self-sustained oscillator. However, this conjecture should be confirmed through the analysis of the time-delayed system (16) and (17) rather than the ordinary differential equations (18) and (19).

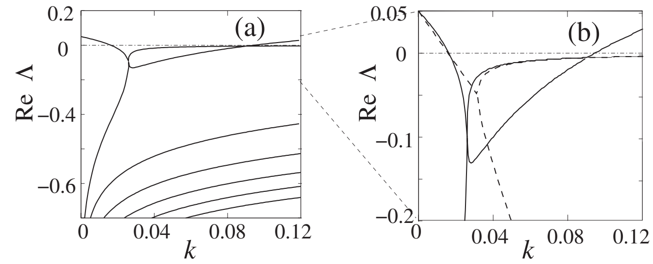

FIG. 2. Real parts of the complex eigenvalues Λ as a function of the control gain k. (a) Numerical solution of the transcendental characteristic equation (22). (b) Enlarged part (the solid line) of (a). The dashed line denotes the approximate eigenvalues obtained analytically from Eq. (23). Parameters: $\lambda_s = \lambda_c = 0.05$; $\omega_0 = 1$; $\omega_c \approx 1$; $\tau = 2\pi/\omega_0$.

The eigenvalues Λ of the fixed point $(r^*, \hat{w}^*) = (r_0, 0)$ of the time-delay system (16), (17) and the approximate ordinary differential equations (18), (19) satisfy the characteristic equation

$$\Lambda^2 - (\lambda_s + \lambda_c)\Lambda + \lambda_s\lambda_c + k(1 - e^{-\Lambda\tau}) = 0 \tag{22}$$

and

$$\Lambda^2 + (k\tau - \lambda_s - \lambda_c)\Lambda + \lambda_s\lambda_c = 0, \tag{23}$$

respectively, where $\lambda_s = \lambda_0$ for $r_0 = 0$ and $\lambda_s = 2\lambda_0$ for $r_0 = \sqrt{\lambda_0}$ and the linearization $f_\pm(r) = \lambda_s(r - r_0)$ around the unstable fixed point r_0 was applied without loss of generality. Note that Eq. (23) follows from Eq. (22) under the approximation $|\mathrm{Re}\,\Lambda|\tau \ll 1$. It is difficult to see analytically the behavior of the solutions of Eq. (22). Whereas, the root loci of Eq. (23) can be seen as considered in [25]: For $k = 0$, the eigenvalues are λ_s and λ_c, which correspond to the free system and free controller, respectively. With the increase of k, they approach each other on the real axis, then collide at $k = (\lambda_s + \lambda_c - 2\sqrt{\lambda_s\lambda_c})/\tau$ and form a complex conjugate pair in the complex plane. At $k = (\lambda_s + \lambda_c)/\tau$, they cross symmetrically into the left half plane (inverse Hopf bifurcation). The condition for stabilization, $k > (\lambda_s + \lambda_c)/\tau$, coincides with Eq. (21). (Note that there exists no upper boundary of control gain k for stabilizing the UPO and USS, the existence of which is often found in time-delay control.) Figure 2 shows the real parts of the complex eigenvalues Λ as a function of control gain k, which was numerically obtained from the

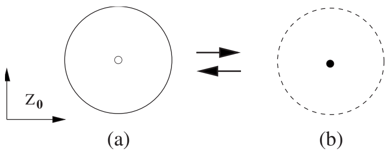

FIG. 3. Conversion of stability in normal forms (complex Z_0) for the (a) supercritical and (b) subcritical Hopf bifurcation by coupling to a supercritical normal form (Z_c). Stable and unstable states are indicated by full lines–circles and broken lines–empty circles, respectively.

exact transcendental equation (22). We see that there exists an interval of control gain k for which the real parts of Λ are negative, so that unstable fixed point r_0 becomes stable. The dashed line in the enlarged part corresponds to the approximate eigenvalues obtained analytically from Eq. (23) and the parameters are chosen in both case as $\lambda_s = \lambda_c = 0.05$, $\omega_0 = 1$, $\omega_c \approx 1, \tau = 2\pi/\omega_0$.

On the other hand, it is obvious that the previously stable fixed point becomes always unstable and is transformed to a saddle point due to the unstable controller: $\lambda_s < 0$ and $\lambda_c > 0$. Thus the stability of the limit cycle and the fixed point is exchanged by time delay coupling of the self-sustained oscillator (Fig. 3).

So far, we have considered a time delay coupling scheme in which the time delay was chosen to be the period of the limit cycle, i.e., $\tau = 2\pi/\omega_0$. Next, a half-period control system with a delay time $\tau/2$ will be considered as follows:

$$\dot{Z}_0(t) = (\pm\lambda_0 + i\omega_0 \mp |Z_0(t)|^2)Z_0(t) + KZ_c(t),$$

$$\dot{Z}_c(t) = (\lambda_c + i\omega_c - |Z_c(t)|^2)Z_c(t) - K[Z_0(t) + Z_0(t - \tau/2)],$$

Note that rather than the difference between the current and delayed state the sum is used as the control signal in the above control system, which corresponds to an additional phase π. We obtain the same equations for the amplitudes and phase difference as Eqs. (8)–(10), i.e.,

$$\dot{r} = f_\pm(r) + Kw \cos \theta,$$

$$\dot{w} = \lambda_c w - K\left[r \cos \theta - r_{\tau/2} \cos\left(\theta + \frac{K\tau w}{2r} \sin \theta\right)\right],$$

$$\dot{\theta} = \Delta\omega + K\left[\left(\frac{r}{w} - \frac{w}{r}\right)\sin \theta - \frac{r_{\tau/2}}{w} \sin\left(\theta + \frac{K\tau w}{2r} \sin \theta\right)\right].$$

This means that the half-period control shows also the conversion of the stability in the Hopf bifurcation system. Moreover, the range of the bifurcation parameter λ_0 available for the conversion can be enlarged since the assumption $|r - r_{\tau/2}|/r \ll 1$ is more easily satisfied than $|r - r_\tau|/r \ll 1$ by decreasing the time delay.

V. TWO DELAY-COUPLED VAN DER POL OSCILLATORS

Above we have considered the normal form of the Hopf bifurcation. The same results on the stability of amplitudes and phases can also be obtained for other systems exhibiting Hopf bifurcations, for example, a delay-coupled Van der Pol oscillator:

$$\ddot{x} \pm (\epsilon_0 - x^2)\dot{x} + \omega_0^2 x = K\dot{u}, \tag{24}$$

$$\ddot{u} - (\epsilon_c - u^2)\dot{u} + \omega_c^2 u = -K(\dot{x} - \dot{x}_\tau), \tag{25}$$

where Eq. (24) shows a sub- and supercritical Hopf bifurcation for the upper and the lower sign, respectively, and ϵ_0 is the bifurcation parameter. The parameter ϵ_c in control system (25) is fixed at a small positive value. Close to the bifurcation point, $\epsilon_0 = 0$, we can apply the averaging method [40] to obtain an approximate solution, which yields the amplitude and phase equations as

$$\dot{r} = \pm r\left(\frac{\epsilon_0}{2} - \frac{r^2}{8}\right) + \frac{K\omega_c w}{2\omega_0} \cos(\varphi - \psi),$$

$$\dot{\varphi} = \omega_0 - \frac{K\omega_c w}{2\omega_0 r} \sin(\varphi - \psi),$$

$$\dot{w} = w\left(\frac{\epsilon_c}{2} - \frac{w^2}{8}\right) - \frac{K\omega_0}{2\omega_c}[r \cos(\varphi - \psi) - r_\tau \cos(\varphi_\tau - \psi)],$$

$$\dot{\psi} = \omega_c - \frac{K\omega_0}{2\omega_c w}[r \sin(\varphi - \psi) - r_\tau \sin(\varphi_\tau - \psi)],$$

where the transformations $x = r\cos\varphi$, $\dot{x} = -\omega_0 r \sin\varphi$, $u = w\cos\psi$ and $\dot{u} = -\omega_c w \sin\psi$ were used. By using an approximation similar to Sec. II, the above equations can be reduced as follows:

$$\dot{r} = f_\pm(r) + \frac{K\omega_c w}{2\omega_0} \cos \theta,$$

$$\dot{w} = \frac{\epsilon_c}{2} w - \frac{K\omega_0}{2\omega_c}\left[r \cos \theta - r_\tau \cos\left(\theta + \frac{K\omega_c \tau w}{2\omega_0 r} \sin \theta\right)\right],$$

$$\dot{\theta} = \Delta\omega + \frac{K}{2}\left[\left(\frac{\omega_0 r}{\omega_c w} - \frac{\omega_c w}{\omega_0 r}\right)\sin \theta - \frac{\omega_0 r_\tau}{\omega_c w} \times \sin\left(\theta + \frac{K\omega_c \tau w}{2\omega_0 r} \sin \theta\right)\right],$$

where $f_\pm(r) = \pm r(4\epsilon_0 - r^2)/8$, $\theta = \varphi - \psi$ and $\Delta\omega = \omega_0 - \omega_c$. These equations are equivalent to Eqs. (8)–(10), which means that all results obtained from the normal form are available also for the Van der Pol oscillators system.

To support the above analytical theory we have performed numerical simulations of the system (24) and (25). The results for the upper sign corresponding to an UPO system and the set of parameters $(\epsilon_0, \epsilon_c, \omega_0, \omega_c) = (0.1, 0.1, 1, 1)$ are shown in Fig. 4. Without control, $K = 0$ ($t < 20$), the system (24) converges to the fixed point at zero and the controller is kept at the fixed point by zero initial condition. With control gain $K = 0.4$ activated at $t = 20$, the controlled system approaches the previously unstable orbit after a transient process [Fig. 4(a)], and the controller and feedback perturbation

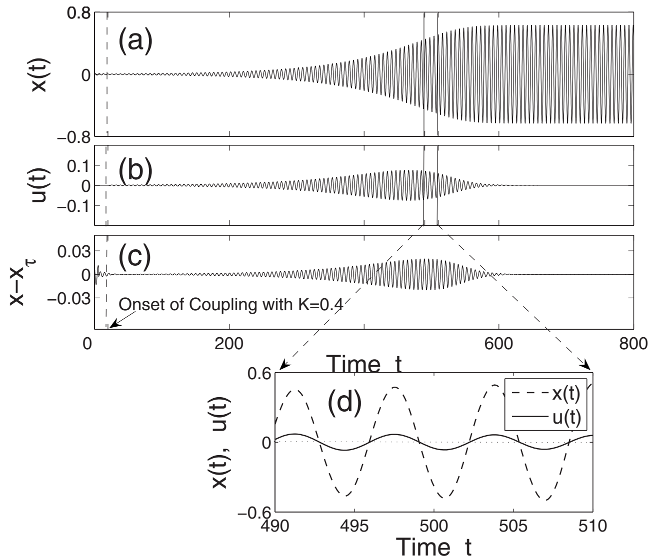

FIG. 4. Results of numerical integration of the delay-differential equations (24) and (25) with the upper sign for $(\epsilon_0,\epsilon_c,\omega_0,\omega_c,\tau)=(0.1,0.1,1,1,2\pi)$. The control perturbation with $K=0.4$ is switched on at $t=20$. The delayed coupling stabilizes an UPO. (a) Dynamics of the x variable, (b) u variable, and (c) perturbation $x-x_\tau$. (d) Enlarged part of (a) and (b) between $t=490$ and $t=510$ where the phase-locking can be seen.

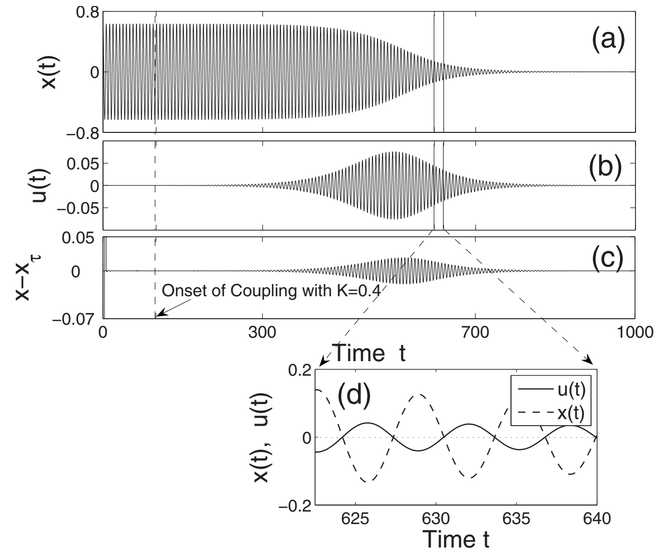

FIG. 5. Same diagrams as in Fig. 4 but for the lower sign in Eqs. (24) and (25). Two delay-coupled Van der Pol oscillators exhibit amplitude death after switching on control at $t=100$ with the same parameters as in Fig. 4 through antiphase locking.

vanish [Figs. 4(a) and 4(b)]. It should be noted that the basin of attraction for stabilizing the UPO includes the whole area inside the UPO in the $(x,\dot{x})$ phase plane. This is in contrast to the result of an unstable controller method [24] in which the basin is limited to an area around the UPO. The enlarged part between $t=490$ and $t=510$ shows the phase locking with the phase difference $\theta=0$ [Fig. 4(d)], which confirms our analytical results.

Figure 5 shows the control of the USS system for the lower sign in Eq. (24). We can see that the two self-sustained oscillations disappear as well as the feedback perturbation after the time delay coupling is activated at $t=100$ with $K=0.4$ [Figs. 5(a)–5(c)]. This is a kind of amplitude death exhibited in the coupled self-sustained oscillators system. Note that the amplitude death occurs with a small coupling strength in contrast to the previous studies [33,34]. The phase difference between two Van der Pol oscillators is maintained with $\theta=\pi$ as shown in Fig. 5(d), which coincides with the theoretically obtained value.

VI. CONCLUSIONS

In this paper, we have proposed a control method with time delayed coupling for converting the stability features of systems close to a Hopf bifurcation. We have considered two delay-coupled normal forms for Hopf bifurcations and demonstrated the phase-locking as well as the conversion of stability of the limit cycle and fixed point, i.e., an interchange of sub- and supercritical Hopf bifurcation. The control system provides us with a unified method for stabilizing both the UPO and USS and it has a larger basin of attraction in comparison with the nonlinear control scheme in [24]. Since the delayed-coupling terms differ from the conventional DFC, the control system is not subject to the odd number limitaion and can be extended to a half-period control as well as a full-period control. We have demonstrated that two self-sustained oscillators with this delayed coupling manifest amplitude death through antiphase synchronization, which holds for rather small coupling strength unlike amplitude death phenomena with a diffusive coupling. The main method and results are applicable to a wide class of systems showing Hopf bifurcations, for example, the Van der Pol oscillator. The analytical theory was supported by numerical simulations of two delay-coupled Van der Pol type systems, which shows good agreement with the theory.

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